Coding Algorithm for Grayscale Images Based on Piecewise Uniform Quantizers

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Abstract. In this paper, a piecewise uniform quantizer for input samples with discrete amplitudes for Laplacian source is designed and analyzed, and its forward adaptation is done. This type of quantizers is very often used in practice for the purpose of compression and coding of already quantized signals. It is shown that the design and the adaptation of quantizers for discrete input samples are different from the design and the adaptation of quantizers for continual input samples. A weighting function for PSQNR (peak signal-to-quantization noise ratio), which is obtained based on probability density function of variance of standard test images is introduced. Experiments are done, applying these quantizers for compression of grayscale images. Experimental results are very well matched to the theoretical results, proving the theory. Adaptive piecewise uniform quantizer designed for discrete input samples gives for 9 to 20 dB higher PSQNR compared to the fixed piecewise uniform quantizer designed for discrete input samples. Also it is shown that the adaptive piecewise uniform quantizer designed for discrete input samples gives higher PSQNR for 1.46 to 3.45 dB compared the adaptive piecewise uniform quantizer designed for continual input samples, which proves that the discrete model is more appropriate for image quantization than continual model.

Keywords: piecewise uniform quantizer, input samples with discrete amplitude, forward adaptation, grayscale image.

1. Introduction

Quantization is one of the main steps in digitalization of analog signals (Jayant and Noll, 1984). Very often, quantization is done in two phases. In the first phase, quantization with large number of levels is done, with the aim of A/D conversion. In such way quantized samples, which have discrete amplitudes, are further quantized in the second phase, using quantizer with much smaller number of levels, with the aim of compression. Quantizers in the first and in the second phase are different, because the quantizer in the first phase has continual input samples (i.e., samples with continual amplitudes) while the quantizer in the second phase has discrete input samples (i.e., samples with discrete amplitudes). Quantizers in the first phase, for the continual input samples, have been analyzed in many papers (Na, 2001, 2008). The aim of this paper is the design and adaptation of the quantizer in the second phase, for discrete input samples. In the paper Savic et al. (2010), the uniform quantizer for discrete input samples was considered.
In this paper, the piecewise uniform quantizer for discrete input samples is considered, for signals with Laplacian distribution. The piecewise uniform quantizer (Nikolic et al., 2011) joins good characteristics of both uniform and nonuniform quantizers: it can achieve $SQNR$ (signal-to-quantization noise ratio) near to $SQNR$ of the nonuniform quantizer but with much less complexity. The piecewise uniform quantizer can be considered as generalized quantized whose special cases are the uniform and the nonuniform quantizer. It will be shown that the design of the quantizer for discrete input samples is different from the design of the quantizer for continual input samples, because discrete samples have bounded amplitudes and therefore the overload distortion does not exist.

The forward adaptation (Nikolic and Peric, 2008) of the piecewise uniform quantizer for discrete input samples is also done in this paper. It is shown that the adaptation of the quantizer for discrete input samples is different from the adaptation of the quantizer for continual input samples. Namely, the adaptation of the quantizer for discrete input samples limited in amplitude should be done only if one condition (that the maximal amplitude of the adaptive quantizer is smaller than the maximal amplitude of the fixed quantizer) is fulfilled, while the adaptation of quantizers for continual input samples can be done without any conditions.

Experiments are done, applying the previously defined fixed and adaptive piecewise uniform quantizers for discrete input samples for grayscale image compression. Three standard test images are analyzed (Lena, Street and Boat). The grayscale image (whose pixels have discrete integer values) is divided into blocks, for each block the mean value is calculated and the difference between the pixel value and the quantized mean value of the block is quantized using the piecewise uniform quantizer for discrete input samples. This quantizer can be applied because the difference has integer (i.e., discrete) values with Laplacian distribution (Jayant, 1984; Salomon, 2007). The standard deviation of the difference, denoted as $\sigma_d$, is calculated. A weighted function for $\sigma_d$, obtained experimentally, based on three previously mentioned test images, is introduced in this paper. As a measure of quality of the reconstructed image, $PSQNR$ (peak signal-to-quantization noise ratio) is used (Sayood, 2006). $PSQNR$ is calculated for all values of $\sigma_d$. The average value of $PSQNR$ is calculated in two ways: by classic averaging (where each value of $\sigma_d$ has the same weight) and by weighted averaging (where the weighting function for $\sigma_d$ is used).

It is shown by experiment that the adaptive piecewise uniform quantizer designed for discrete input samples gives for 9 to 20 dB higher $PSQNR$ compared to the fixed piecewise quantizer designed for discrete input samples. Also, the adaptive quantizer designed for discrete input samples gives for 1.46 to 3.45 dB higher $PSQNR$ compared to the adaptive quantizer designed for continual input samples. It proves that the discrete model is more appropriate for image quantization than continual model.

It is shown that experimentally obtained $PSQNR$ (averaged using the weighted function for $\sigma_d$) is matched very well with $PSQNR$ theoretically obtained. In this way, theoretical analysis for the fixed and the adaptive piecewise uniform quantizers for discrete input samples is proved. Also, the validity of the introducing of the weighted function for $\sigma_d$ is proved. Using the adaptive piecewise quantizer, the near lossless compression is achieved.
This piecewise uniform quantizer for discrete input samples can be used, besides for the image compression, also for compression of all other signals with Laplacian distribution.

2. Construction of Fixed Piecewise Uniform Quantizer for Discrete Input

Quantization is often done in two phases. In the first phase, samples with continual amplitude are quantized with the uniform quantizer \( Q_0 \) with \( N_0 \) output levels \( X = \{ x_1, \ldots, x_{N_0} \} \). Continual amplitudes of the input samples are described with some probability density function (pdf), denoted with \( p(x) \). In this paper, we will use Laplacian pdf, which is defined as \( p(x) = \left( \frac{1}{\sqrt{2\sigma}} \right) \exp\left( -\frac{\sqrt{2|x|}}{\sigma} \right) \), where \( \sigma \) is the standard deviation of the random variable \( x \). The maximal amplitude of the uniform quantizer \( Q_0 \) is denoted as \( x_{\text{max}} \), which depends on the range of the input signal. The design of the quantizer \( Q_0 \) can be described in another way: firstly, the uniform quantizer for the unit standard deviation \( \sigma = 1 \) is designed with the maximal amplitude \( x_{\text{max}}^0 \) [Jayant] and after that the denormalization is done, by dividing all thresholds and representation levels with \( \sqrt{x_{\text{max}}^0} \), with the aim to map the range \([-x_{\text{max}}^0, x_{\text{max}}^0]\) to the range \([-x_{\text{max}}, x_{\text{max}}]\). In the second step, output samples from the quantizer \( Q_0 \) (which have discrete amplitudes) are further quantized with the second quantizer \( Q \) with \( N \) levels, \( N < N_0 \). Input samples of the quantizer \( Q \) can take \( N_0 \) discrete values, which are equal to the output levels of the quantizer \( Q_0 \), defined with the set \( X \). Probabilities of these discrete levels for Laplacian distribution are 
\[
P(x_i) = p(x_i)\triangle_0 = \left( \frac{1}{\sqrt{2\sigma}} \right) \exp\left( -\frac{\sqrt{2|x|}}{\sigma} \right) \triangle_0, \quad i = 2, \ldots, N_0 - 1 \quad \text{and} \quad P(x_1) = P(x_{N_0}) = \left( \frac{1}{2} \right) \exp\left( -\frac{\sqrt{2x_{\text{max}}}}{\sigma} \right).
\]

The aim of this paper is design of the quantizer \( Q \). Since this quantizer is used for quantization of the samples with discrete amplitudes, its design is different from the design of the quantizers with continual amplitude samples. In this paper quantizer \( Q \) will be realized as piecewise uniform quantizer with \( N \) levels grouped in \( L \) regions. The piecewise uniform quantizer can be considered as the generalized quantizer, since for \( L = N \) the nonuniform quantizer is obtained and for \( L = 1 \) the uniform quantizer is obtained.

Each region has \( M = N/L \) uniform output levels. \( N_i \) denotes the number of the input levels from the set \( X \) which belong to the \( i \)th region. Then, it holds that \( N_0 = \sum_{i=1}^{L} N_i \), \( \mu_i \) is the parameter of the \( i \)th region which represents the number of input levels that are mapped to one output level, in the \( i \)th region, \( \varphi_i, \ i = 0, \ldots, L \), denote the boundaries between regions. The design of the piecewise uniform quantizer \( Q \) will be done in the following way. Firstly, the piecewise uniform quantizer with \( N \) levels for the unit standard deviation \( \sigma = 1 \) is designed. The boundaries between the regions, denoted as \( \varphi_{i}^k, \ i = 0, \ldots, L \), are obtained as \( \varphi_{i}^k = t_{M} \), where \( t_{j}, \ j = 0, \ldots, N \) represent the thresholds of the optimal companding quantizer with \( N \) levels for \( \sigma = 1 \) which are given with expression (4) in the paper Peric et al. (2009), which was derived for the optimal compression function defined in Judell and Scharf (1986), Na (2004). In this way, the following expressions for \( \varphi_{i}^k \) are obtained:
\[ \varphi_{\sigma=1} = \frac{3}{\sqrt{2}} \log \left( \frac{2iM + (N - 2iM) \exp(-\sqrt{2}t_{\sigma=1} t_{\text{max}}(N))}{N} \right), \quad 0 \leq i \leq L/2, \]

\[ \varphi_{\sigma=1} = \frac{3}{\sqrt{2}} \log \left( \frac{N}{2N - 2iM + (2iM - N) \exp(-\sqrt{2}t_{\sigma=1} t_{\text{max}}(N))} \right), \quad L/2 < i \leq L. \]

\( t_{\sigma=1} t_{\text{max}}(N) \) denotes the maximal amplitude of the optimal companding quantizer for the unit variance, which values can be found in Peric et al. (2009). The denormalization, by dividing \( \varphi_{\sigma=1} \) with \( \Delta_0 \) is done. The range of the quantizer obtained in this way \([-t_{\text{max}}(N), t_{\text{max}}(N)]\), \( t_{\text{max}}(N) = t_{\sigma=1} t_{\text{max}}(N)/\Delta_0 \), is different from the range \([-x_{\text{max}}, x_{\text{max}}]\), since \( t_{\sigma=1} t_{\text{max}}(N) \neq x_{\sigma=1} x_{\text{max}} \). If \( t_{\text{max}}(N) < x_{\text{max}} \), the overload distortion will exist. If \( t_{\text{max}}(N) > x_{\text{max}} \), the range \([x_{\text{max}}, t_{\text{max}}(N)]\) will be unused, which leads to higher granular distortion. To avoid these negative effects, the corrections of the quantizer range will be done by dividing the boundaries between the regions with \( t_{\max}/x_{\text{max}} \). In this way, the boundaries between the regions \( \varphi_i \) are obtained. The range of the quantizer is \([-x_{\text{max}}, x_{\text{max}}]\).

\( d_i, i = 1, \ldots, L \) denotes the width of the \( i \)th region of the fixed piecewise uniform quantizer \( Q \) and it can be calculated as

\[ d_i = \frac{\varphi_i - \varphi_{i-1}}{M}. \]

The output levels of the quantizer \( Q \), denoted with \( y_{ij}, i = 1, \ldots, L; j = 1, \ldots, M \) (\( i \) represents the region where the output level belongs and \( j \) represents the number of the output level within that region) can be calculated as

\[ y_{ij} = t_{i-1} + \left( \frac{2j - 1}{2} \right) d_i. \]

During the quantization process, an irreversible error is made, which is defined with distortion \( D \). Since input samples have discrete amplitudes which are limited with \( x_{\text{max}} \), the fixed quantizer \( Q \) has only the granular distortion \( D_g \), which is defined as:

\[ D = D_g = \sum_{i=1}^{L} \sum_{j=1}^{M} \mu_i \sum_{k=1}^{\mu_i} (x_{ijk} - y_{ij})^2 P(x_{ijk}), \]

where \( x_{ijk} \in X \) is one of \( \mu_i \) input levels which are mapped to the output level \( y_{ij} \).

The quality of the quantized signal is usually defined with the signal-to-quantization noise ratio \( SQNR \), which is defined in the following way:

\[ SQNR [\text{dB}] = 10 \log_{10} \left( \frac{\sigma^2}{D_g} \right), \]

where \( \sigma^2 \) is a variance of the input signal with discrete amplitudes.
Table 1

<table>
<thead>
<tr>
<th>SQNR [dB]</th>
<th>N = 16, ( t^{\sigma=1}_{\text{max}}(N) = 6.01 )</th>
<th>N = 32, ( t^{\sigma=1}_{\text{max}}(N) = 7.41 )</th>
<th>N = 64, ( t^{\sigma=1}_{\text{max}}(N) = 8.85 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>L = 2</td>
<td>10.90</td>
<td>16.427</td>
<td>23.071</td>
</tr>
<tr>
<td>L = 4</td>
<td>14.962</td>
<td>20.90</td>
<td>26.098</td>
</tr>
<tr>
<td>L = 8</td>
<td>16.652</td>
<td>22.826</td>
<td>27.828</td>
</tr>
<tr>
<td>L = 16</td>
<td>17.417</td>
<td>23.454</td>
<td>29.069</td>
</tr>
</tbody>
</table>

SQNR\(_{\text{opt}}\) [dB] 18.080 23.840 29.727

In Table 1, values of SQNR for piecewise uniform quantizer are given, for different values of \( N \) and \( L \), for the input signal with Laplacian distribution with unit variance \((\sigma^2 = 1)\). Input samples have discrete amplitudes which have been already quantized with the uniform quantizer with \( N_0 = 512 \) levels and \( x^{\sigma=1}_{\text{max}} = 7.9 \). In the last row of Table 1, values of SQNR for the optimal nonuniform quantizer for samples with continuous amplitudes is given, denoted with SQNR\(_{\text{opt}}\). These values are taken from the paper Peric et al. (2009). Based on the results shown in Table 1, we can conclude that if the number of regions \( L (L \leq N) \) increases, performances of piecewise uniform quantizer for the discrete input became more closer to the performances of the optimal nonuniform quantizer for the continual input.

3. Design of the Adaptive Piecewise Uniform Quantizer for Discrete Input Samples

In this section, the forward adaptation of the piecewise uniform quantizer for discrete input samples will be done. The adaptation is done on the frame-by-frame (or, block-by-block for the image compression) basis. The standard deviation \( \sigma \) of the frame is found. Quantized standard deviation of the frame \( \hat{\sigma} \) is transmitted to the receiver. The maximal amplitude of the adaptive quantizer before denormalization, denoted as \( t^{\text{adapt}}_{\text{max}} \), is defined as:

\[
  t^{\text{adapt}}_{\text{max}} = k\hat{\sigma},
\]

where \( k \) is the loading factor of the adaptive quantizer, whose optimal value is found numerically, as it will be described in the Section 5.2. The boundaries between the regions \( \varphi^{\text{adapt}}_i \) of the piecewise uniform adaptive quantizer is defined with the following expressions:

\[
  \varphi^{\text{adapt}}_i = \frac{3\hat{\sigma}}{\sqrt{2} \Delta_0} \log \left( \frac{2iM + (N - 2iM) \exp(-\frac{3\hat{\sigma}}{4} t^{\text{adapt}}_{\text{max}})}{N} \right),
\]

\[
  0 \leq i \leq L/2,
\]
\[ \varphi_i^{\text{adapt}} = \frac{3\sigma}{\sqrt{2} \Delta_0} \log \left( \frac{N}{2N - 2iM + (2iM - N)\exp\left(-\frac{\sqrt{2}}{\sigma} \Delta_0^{\text{adapt}}\right)} \right), \]

\[ L/2 < i \leq L. \]  

Division with \( \Delta_0 \) in the previous expressions is done due to denormalization. The maximal amplitude of the adaptive piecewise uniform quantizer is \( x^{\text{adapt}}_{\text{max}} = x^{\text{adapt}}_{\text{max}} / \Delta_0 \) (after denormalization). \( d_i^{\text{adapt}}, i = 1, \ldots, L \), which denotes the width of the \( i \)th region of the piecewise uniform adaptive quantizer, is defined as:

\[ d_i^{\text{adapt}} = \frac{\varphi_i^{\text{adapt}} - \varphi_{i-1}^{\text{adapt}}}{M}. \]

The output levels of this quantizer, denoted as \( y_{ij}^{\text{adapt}}, i = 1, \ldots, L; j = 1, \ldots, M \), are defined as:

\[ y_{ij}^{\text{adapt}} = \varphi_{i-1}^{\text{adapt}} + \left( \frac{2j - 1}{2} \right) d_i^{\text{adapt}}. \]

Using adaptation, we adjust the amplitude range of the quantizer to the variance of the input signal. In Fig. 1, the range \( I = (-x_{\text{max}}, x_{\text{max}}) \) of the uniform quantizer \( Q_0 \), which precedes to the adaptive piecewise uniform quantizer, is shown firstly. Recall that input samples of the adaptive quantizer are bounded in amplitude with \( x_{\text{max}} \). Also, in Fig. 1, the range \( I^{\text{adapt}} = (-x^{\text{adapt}}_{\text{max}}, x^{\text{adapt}}_{\text{max}}) \) of the adaptive piecewise uniform quantizer is shown for two cases: when \( x^{\text{max}}_{\text{adapt}} < x_{\text{max}} \) (i.e., \( I^{\text{adapt}} \) becomes narrower than \( I \)) and when \( x^{\text{max}}_{\text{adapt}} > x_{\text{max}} \) (i.e., \( I^{\text{adapt}} \) becomes wider than \( I \)). The case when \( x^{\text{max}}_{\text{adapt}} > x_{\text{max}} \) will be deeply analyzed. In this case, since the input samples are amplitudely bounded by \( x_{\text{max}} \), the representation levels in the range \((-x^{\text{adapt}}_{\text{max}}, -x_{\text{max}}) \cup (x_{\text{max}}, x^{\text{adapt}}_{\text{max}}) \) is unused. But, as it was said earlier, input samples are bounded in amplitude with \( x_{\text{max}} \). Therefore, not all \( N \) representation levels are used, but only \( N_1 \) (\( N_1 \) is some number smaller than \( N \)). Because of that, higher distortion (i.e., lower SQNR) is obtained, compared to the

![Fig. 1. The range of the fixed and the forward adaptive quantizers.](image)
fixed quantizer (without adaptation). We can conclude that if adaptation is done when \( x_{\text{adapt}}^{\text{max}} > x_{\text{max}} \), worse performances are obtained. Therefore, adaptation should not be done in this case. We can introduce the following rule: adaptation should be done only when \( x_{\text{adapt}}^{\text{max}} < x_{\text{max}} \). If this condition is not fulfilled, the fixed quantizer should be used.

This is the main difference between adaptive quantizers for continual and for discrete input samples, because for continual input samples adaptation gives good results in both cases: \( x_{\text{adapt}}^{\text{max}} < x_{\text{max}} \) and \( x_{\text{adapt}}^{\text{max}} > x_{\text{max}} \).

For the adaptive quantizer, input levels can take values from the range \((-x_{\text{max}}^{\text{adapt}}, x_{\text{max}}^{\text{adapt}})\) and the output values can take values from the range \((-x_{\text{adapt}}^{\text{max}}, x_{\text{adapt}}^{\text{max}})\), \( x_{\text{adapt}}^{\text{max}} < x_{\text{max}} \). The input levels from the range \( (x_{\text{adapt}}^{\text{max}}, x_{\text{max}}^{\text{adapt}}) \), whose number is denoted as \( \mu_{L+1} \), are mapped into the last output level \( y_{L,M} \), which leads to the existence of the overload distortion \( D_{ov} \). Therefore, for the adaptive quantizer, we have the granular distortion given with the expression (5) and the overload distortion given with the following expression:

\[
D_{ov} = 2 \sum_{k=1}^{\mu_{L+1}} (x_{LMk} - y_{LM})^2 P(x_{LMk}).
\] (12)

The total distortion is equal to the sum of these two distortions, i.e.,:

\[
D = D_g + D_{ov}.
\] (13)

4. Application of the Previous Models on the Grayscale Images

In this section, a weighting function for \( PSQNR \) calculation is introduced and an algorithm for image processing is presented.

4.1. Weighting Function

As a measure of the quality of the piecewise uniform quantizer \( Q \) (fixed or adaptive) we will use the peak signal-to-quantization noise ratio for the block \( B \) of the original image pixels, defined as

\[
PSQNR \ [\text{dB}] = 10 \log_{10} \left( \frac{x_{\text{max}}^2}{D} \right) = 10 \log_{10} \left( \frac{x_{\text{max}}^2}{D_d} \right) .
\] (14)

Since the distortion \( D_d \) depends on \( \sigma_d \), it follows that \( PSQNR \ [\text{dB}] \) also depend on \( \sigma_d \) which can take values from 1 to \( x_{\text{max}} \) (\( x_{\text{max}} = 255 \)). Actually, \( \sigma_d \) also can take value 0, but this case will not be considered since if \( \sigma_d = 0 \) then block \( B_d \) does not contain any information (in this case \( x_{av} \) carries all information about block \( B \)). \( \sigma_d \) can be written in the logarithmic domain as \( \sigma_d \ [\text{dB}] = \sigma_d \ [\text{dB}] = 20 \log_{10} \frac{\sigma_d}{\sigma_0} \), where \( \sigma_0 \) is some referent standard deviation. Without losing a generality, we can take that \( \sigma_0 = 255 \). Then \( \sigma_d \ [\text{dB}] \) can take values from 20\( \log_{10} \frac{1}{255} \) = -48.13 dB to 20\( \log_{10} \frac{255}{255} \) = 0 dB. As a measure of the quality of the reconstructed block of the image, we will use the average \( PSQNR \) in the range \([-48.13 \text{ dB}, 0 \text{ dB}] \) of \( \sigma_d \ [\text{dB}] \). The averaging will be done in two ways:
1. All values of $\sigma_d$ [dB] in that range have the same weight. In that way we obtained $PSQNR_{av}$:

$$PSQNR_{av} = \frac{1}{255} \sum_{\sigma_d=1}^{255} PSQNR(\tilde{\sigma}_d) \text{ [dB]}.$$  \hspace{1cm} (15)

2. Since all of the input samples do not occur with the same probabilities we introduce weighting function for $\sigma_d$ [dB]. This weighting function is calculated based on the three standard test images (Lena, Boat and Airplane) and it is shown in Fig. 2a in the linear domain and in Fig. 2b in the logarithmic domain. In this way $PSQNR_{wav}$ is obtained.

$$PSQNR_{wav} = \sum_{\sigma_d=1}^{255} w(\tilde{\sigma}_d)PSQNR(\tilde{\sigma}_d) \text{ [dB]}.$$  \hspace{1cm} (16)

4.2. An Algorithm for the Image Processing

In this section, we will explain the application of the previously described quantizers (fixed and adaptive) on grayscale images. Pixels of the image can take integer values from 0 to $x_{max}$. We consider the case whit $x_{max} = 255$ (each pixel is represented with 8 bits). The algorithm for the image processing starts with dividing the image into blocks, whose size is $m \times m$ (we will use $m = 4$). Each block is processed separately. Now, an algorithm will be described, where the fixed piecewise uniform quantizer is used for the processing of one arbitrary block, which is denoted with B and whose pixels are denoted with $x_{i,j}$, $i = 1, \ldots, m$; $j = 1, \ldots, m$.

1. The average value $x_{av}$ of the block B is found, this value is quantized and obtained quantized average value $\tilde{x}_{av}$ is transmitted to the receiver.
2. The difference block $B_d$, which size is $m \times m$ is formed. Elements of that block, denoted with $d_{i,j}$, $i = 1, \ldots, m; j = 1, \ldots, m$ are obtained as the difference of the pixel values of the block $B$ and the quantized mean value $\hat{x}_{av}$, i.e.,

$$d_{i,j} = x_{i,j} - \hat{x}_{av}. \quad (17)$$

Elements $d_{i,j}$ of the block $B_d$ can take integer values from the interval $[-x_{max}, x_{max}]$. $d_{i,j}$ are integer because they are obtained as a difference between two integer values. Elements of the block $B_d$ have the Laplacian distribution (Jayant, 1984).

3. Since the elements of the block $B_d$ are integers (i.e., they have discrete amplitudes), fixed quantizer designed for discrete input should be used. Quantized elements of the block $B_d$, denoted with $\hat{d}_{i,j}$ are binary coded with $R$ bits ($\log_2 N$) and transmitted to the receiver. The distortion which is made by quantization of the elements $d_{i,j}$ can be expressed as:

$$D_d = \frac{1}{m \times m} \sum_{i=1}^{m} \sum_{j=1}^{m} (d_{i,j} - \hat{d}_{i,j})^2. \quad (18)$$

4. In the receiver, after the reception of $\hat{x}_{av}$ and $\hat{d}_{i,j}$, the reconstruction of the pixels of the original image is done as

$$\hat{x}_{i,j} = \hat{d}_{i,j} + \hat{x}_{av}. \quad (19)$$

The reconstructed block of the image, denoted as $\hat{B}$ consists of the reconstructed pixels $\hat{x}_{i,j}$. The difference between the original block $B$ and the reconstructed block $\hat{B}$ is measured by distortion $D = \frac{1}{m \times m} \sum_{i=1}^{m} \sum_{j=1}^{m} (x_{i,j} - \hat{x}_{i,j})^2$. Based on (14), (15) and (16), it follows that

$$D = \frac{1}{m \times m} \sum_{i=1}^{m} \sum_{j=1}^{m} (x_{i,j} - \hat{x}_{i,j})^2 = \frac{1}{m \times m} \sum_{i=1}^{m} \sum_{j=1}^{m} (d_{i,j} - \hat{d}_{i,j})^2 = D_d. \quad (20)$$

For the adaptive quantizer, the previous algorithm with the following changes is used. The Step 2 is extended with the Step 2.1, where the standard deviation $\sigma_d$ of the block $B_d$ is calculated as $\sigma_d = \sqrt{\frac{1}{m \times m} \sum_{i=1}^{m} \sum_{j=1}^{m} (x_{i,j} - \hat{x}_{av})^2}$ and transmitted to the receiver. $[x]$ denotes the nearest integer for $x$. After that, in the Step 3, the adaptive piecewise uniform quantizer, designed for $\sigma_d$, is used instead of the fixed quantizer.

In Table 2, lower ($x_{i}^{\text{low}}$) and upper ($x_{i}^{\text{up}}$) boundaries of intervals of the fixed and the adaptive quantizers used in the Step 3 for quantization of the block $B_d$ are shown, for $N = 16$, $L = 16$ and for different values of $\sigma_d$. Due to the symmetry, only the positive
Table 2

Boundaries of the intervals (thresholds) for the fixed and the adaptive piecewise uniform quantizer (designed for discrete input samples), applied on the image compression, for \(N = 16\), \(L = 16\) and for different values of \(\sigma_d\).

<table>
<thead>
<tr>
<th>(i)</th>
<th>Fixed quantizer</th>
<th>Adaptive quantizer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(x_i^{\text{low}})</td>
<td>(x_i^{\text{up}})</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>17</td>
<td>32</td>
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<tr>
<td>3</td>
<td>33</td>
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<tr>
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<td>49</td>
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</tr>
<tr>
<td>5</td>
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<td>204</td>
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<tr>
<td>8</td>
<td>205</td>
<td>255</td>
</tr>
</tbody>
</table>

Table 3

Boundaries of the intervals (thresholds) for the fixed and the adaptive piecewise uniform quantizer (designed for continual input samples), applied on the image compression, for \(N = 16\), \(L = 16\) and for different values of \(\sigma_d\).

<table>
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<tr>
<td></td>
<td>(x_i^{\text{low}})</td>
<td>(x_i^{\text{up}})</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>19</td>
<td>30</td>
</tr>
<tr>
<td>4</td>
<td>31</td>
<td>43</td>
</tr>
<tr>
<td>5</td>
<td>44</td>
<td>61</td>
</tr>
<tr>
<td>6</td>
<td>62</td>
<td>84</td>
</tr>
<tr>
<td>7</td>
<td>85</td>
<td>119</td>
</tr>
<tr>
<td>8</td>
<td>120</td>
<td>194</td>
</tr>
</tbody>
</table>

The widths of the intervals \(d_i = x_i^{\text{up}} - x_i^{\text{low}}\) are given. Since the pixels of the images are integers, boundaries of the intervals are also integers. For some intervals, lower and upper boundaries are the same. These intervals have only one representation level, which is equal to the boundaries and the widths of these intervals \(d_i\) are equal to 0. This case is specific for the discrete source; it does not exist for the continual source.

In Table 3, decision thresholds \(x_i^{\text{low}}\) and \(x_i^{\text{up}}\) and widths of the intervals \(d_i\) are given, for fixed and adaptive piecewise uniform quantizers designed for the continual input samples. Due to the symmetry, only the positive parts of the quantizers are shown. Now, we will consider the case when \(\sigma_d = 100\). From Table 3, we can see that the range of the
adaptive quantizer designed for continual input is $[-608, 608]$, which is much wider than the range of the possible input values of discrete signal $[-255, 255]$. It means that the levels in the range $[-608, -255] \cup [255, 608]$ are unused, i.e., the number of levels $N_1$ in the range $[-255, 255]$ which are effectively used is smaller than total number of levels $N$. This is shown in Fig. 3. In general, the same phenomenon occurs for large values of $\sigma_d$. On the other side, the range of the adaptive quantizer for discrete input is limited with the maximal input value (in our case $[-255, 255]$) and therefore all $N$ representation levels are effectively used. Due to unused levels, adaptive quantizer designed for continual input samples has smaller $PSQNR$ compared to the adaptive quantizer designed for discrete input samples, which will be experimentally proved in the next section.

5. Numerical and Experimental Results

In this section numerical and experimental results are presented for fixed and adaptive piecewise uniform quantizer for discrete and continual input samples.

5.1. Numerical Results for the Fixed Piecewise Uniform Quantizer

In Fig. 4, theoretically obtained (using the expression (5) for the distortion) dependences of $PSQNR$ on $\sigma_d$ [dB] are shown for $N = 16$ and $N = 32$, for the fixed piecewise uniform quantizer for discrete input samples. Two average values of $PSQNR$ ($PSQNR_{av}$ and $PSQNR_{wav}$) are calculated in the way described in the Section 4.1 and these values are given in Fig. 4. We can see that these values are quite different.

In Table 4, experimental results, obtained by applying the fixed piecewise uniform quantizer for discrete input samples for image compression (in Step 3 of the algorithm in Section 4.1) of three previously mentioned standard test images, for different values of $N$ and $L$ are shown. Values in Table 4 are averaged values for these three images. We can see that experimentally obtained values of $PSQNR$ (from Table 4) are matched very well to theoretically obtained values of $PSQNR_{wav}$ (in Fig. 4), while they are quite different.
Fig. 4. PSQNR of fixed piecewise uniform quantizer for discrete input for: (a) $N = 16$ and (b) $N = 32$.

Table 4

<table>
<thead>
<tr>
<th>$N$ = 16</th>
<th>$N$ = 32</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L = 2$</td>
<td>$L = 2$</td>
</tr>
<tr>
<td>$L = 4$</td>
<td>$L = 4$</td>
</tr>
<tr>
<td>$L = 8$</td>
<td>$L = 8$</td>
</tr>
</tbody>
</table>

| PSQNR [dB] | 25.358   | 33.037   | 35.817   | 33.272   | 40.058   | 42.380   | 42.640   |

from values of $PSQNR_{av}$ (in Fig. 4). It proves that the theory obtained in Section 2 is correct. Also, it justifies the introduction of the weighting function for $\sigma_d$ and shows that $PSQNR_{wav}$ should be used instead of $PSQNR_{av}$.

Experimental results for the fixed piecewise uniform quantizer designed for continual input samples are similar with the results in Table 4, since the value of $\sigma_d$ which was used ($\sigma_d = 32$) is adjusted to the test images.

5.2. Numerical Results for the Adaptive Piecewise Uniform Quantizer

In this section, numerical results for the adaptive piecewise uniform quantizer for discrete input samples are given. In Fig. 5, theoretically obtained (using the expressions (5), (12) and (13) for the distortion) dependences of $PSQNR$ on $\sigma_d$ [dB] are shown for $N = 16$ and $N = 32$. $PSQNR_{wav}$ is also shown in this figure. $PSQNR_{av}$ is not considered, due to the conclusion in the previous section. The optimal values of the parameter $k$, which are numerically found to maximize $PSQNR_{wav}$, are also given in Fig. 5.

From Fig. 5 we can see an interesting effect: $PSQNR$ for adaptive quantizers increases for small $\sigma_d$ [dB]. Now, we will explain this effect considering one example with parameters $N_0 = 256$, $N = 32$ and $L = 8$. For the fixed quantizer, we choose 32 output levels from the set of 256 input levels, i.e., matching between input and output levels is 12.5%. For the adaptive quantizer, if $\sigma_d$ is small, $x_{\text{max}}^{\text{adapt}}$ also will be small. We consider the case:
Fig. 5. PSQNR of adaptive piecewise uniform quantizer for discrete input for: (a) $N = 16$ and (b) $N = 32$.

Table 5
Experimental results for the adaptive piecewise uniform quantizer designed for discrete and continual input samples

<table>
<thead>
<tr>
<th></th>
<th>$N = 16$</th>
<th></th>
<th>$N = 32$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$L = 2$</td>
<td>$L = 4$</td>
<td>$L = 8$</td>
</tr>
<tr>
<td>PSQNR [dB]</td>
<td>45.029</td>
<td>45.483</td>
<td>45.785</td>
</tr>
<tr>
<td>PSQNR&lt;sub&gt;cont&lt;/sub&gt; [dB]</td>
<td>41.929</td>
<td>42.883</td>
<td>44.345</td>
</tr>
</tbody>
</table>

$x_{\text{adapt}}^\text{max} = \frac{1}{4} \cdot x_{\text{max}}$. Therefore, inside the range of the adaptive quantizer $(-x_{\text{max}}^\text{adapt}, x_{\text{max}}^\text{adapt})$ there are $N_0/4 = 64$ input levels and among them we choose $N = 32$ output levels. So, for the adaptive quantizer, matching between input and output levels is 50%. If some input level is equal to some output level, distortion for that input level is zero. Therefore, if percentage of matching between input and output levels increases then distortion decreases and then $PSQNR$ increases. It is clear from the above example that matching percentage increases when $\sigma_d$ decreases.

In Table 5, experimental results for the adaptive piecewise uniform quantizer for discrete input samples are given, for different values of $N$ and $L$, obtained by applying this quantizer in Step 3 of the algorithm in Section 4.2. Values in Table 5 are averaged values for standard test images (Lena, Street and Boat). We can see that experimentally obtained values of $PSQNR$ in Table 5 are matched very well to theoretically obtained values of $PSQNR_{\text{wav}}$ in Fig. 5. It proves the theory from Section 3.
PSQNR values in Table 5 are very high, which means that high quality reconstructed images are obtained, i.e., near lossless compression is achieved using this adaptive quantizer.

In the aim of comparison, the average values of PSQNR (denoted as \(\text{PSQNR}_{\text{cont}}\)) obtained by application of the adaptive piecewise uniform quantizer designed for continual input samples of three above mentioned standard test images are also given in Table 5. We can see that the adaptive quantizer designed for discrete input samples gives higher PSQNR for 1.46 to 3.45 dB compared the adaptive quantizer designed for continual input samples, because of the reason explained at the end of Section 4.2.

6. Conclusion

Design of fixed and adaptive piecewise uniform quantizers for discrete input samples with Laplacian distribution was considered in this paper. These quantizers have a great practical importance since they are used for compression of already quantized signals. Piecewise uniform quantizer was analyzed since it could be considered as generalized quantizer. It was shown that the design of quantizers for discrete input samples is different from the design of quantizers for continual input samples. Since discrete input samples are limited in amplitude, the overload distortion does not exist. Adaptation of these quantizers should be done only under one condition: the maximal amplitude of the adaptive quantizer should be smaller than the maximal amplitude of the fixed quantizer. Also, for the adaptive quantizer, PSQNR increases for small input variances. An algorithm for compression of grayscale images was presented, applying these quantizers for discrete input samples. A weighting function for the standard deviation of the difference of the pixel value and the quantized mean value of the block which pixels belong was introduced. Average PSQNR was calculated in two ways: by classic averaging (without using weighting function) and by weighted averaging (using weighting function). Experiments were done, applying these fixed and adaptive quantizers for compression of three test grayscale images (Lena, Street and Boat). It was shown that experimentally obtained PSQNR was matched very well with theoretically obtained weighted PSQNR. In this way, developed theory was proved. Also, it was shown that the average PSQNR obtained by using the weighting function should be used instead of the average PSQNR obtained without using the weighting function. We have compared adaptive piecewise uniform quantizer designed for discrete input samples with fixed piecewise uniform quantizer designed for discrete input samples, in order to show that higher PSQNR of 9 to 20 dB is obtained. Also it is shown that the adaptive piecewise uniform quantizer designed for discrete input samples gives higher PSQNR for 1.46 to 3.45 dB compared the adaptive piecewise uniform quantizer designed for continual input samples. Capitalizing on this, the conclusion arises, that the discrete model is more appropriate for image quantization than continual model. With the adaptive piecewise uniform quantizer, the near lossless compression was achieved.
Coding Algorithm for Grayscale Images Based on Piecewise Uniform Quantizers

References


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Kodavimo algoritmas pustoniams vaizdams, grindžiamas atkarpomis tiesiniu, pastoviuoju dydžiu keitikliu

Milan S. SAVIĆ, Zoran H. PERIĆ, Milan R. DINČIĆ

Šiame straipsnyje atkarpomis tiesinis, pastovusis dydžių keitiklis yra sudarytas ir išanalizuotas Laplaso šaltinio įėjimo atskaitoms su diskrečiosiomis amplitudėmis. Gauta tolimesnė adaptacija. Tokio tipo keitikliai yra labai dažnai taikomi praktikoje jau kvantuotoje signalų glaudinimu bei kodavimu. Parodyta, kad šių keitiklių sudarymas bei jų adaptacija diskrečioms įėjimo atskaitoms skiriasi nuo keitiklių sudarymo bei jų adaptacijos tolydinėms įėjimo atskaitoms. Pateikta svorinė PSQNR (didžiausios signalo ir kvantavimo triukšmo reikšmę) svorinė funkcija, kuri gauta taikant standartinio testavimo nuokrypio tikimybini tanki. Atlikti eksperimentai, šiais keitikliais glaudinant pustonių vaizdus. Eksperimentų rezultatai patvirtina teorinius rezultatus. Adaptyvusis atkarpomis tiesinis, pastovusis keitiklis, sukurtas diskrečioms įėjimo atskaitoms, duoda nuo 9 iki 20 dB aukštesnį PSQNR lyginant su fiksuotuojatarpomis tiesiniu keitikliu, sudarytu diskrečiosioms įėjimo atskaitoms. Taip pat yra parodyta, kad adaptyvusis atkarpomis tiesinis, pastovusis keitiklis, sudarytas diskrečiosioms įėjimo atskaitoms duoda nuo 1.46 iki 3.45 aukštesnį PSQNR, nei adaptyvusis atkarpomis tiesinis keitiklis, sudarytas tolydinėms įėjimo atskaitoms. Šis faktas išrodo, kad diskreťusis modelis esti labiau tinkamas vaizdo kvantavimui nei tolydinis modelis.
1. All values of $\sigma_d$ [dB] in that range have the same weight. In that way we obtained $PSQNR_{av}$:

$$PSQNR_{av} = \frac{1}{255} \sum_{\sigma_{d_i}=1}^{255} PSQNR(\hat{\sigma}_{d_i}) \ [dB].$$  \hspace{1cm} (15)

2. Since all of the input samples do not occur with the same probabilities we introduce weighting function for $\sigma_d$ [dB]. This weighting function is calculated based on the three standard test images (Lena, Boat and Airplane) and it is shown in Fig. 2a in the linear domain and in Fig. 2b in the logarithmic domain. In this way $PSQNR_{wav}$ is obtained.

$$PSQNR_{wav} = \sum_{\sigma_{d_i}=1}^{255} w(\hat{\sigma}_{d_i}) PSQNR(\hat{\sigma}_{d_i}) \ [dB].$$  \hspace{1cm} (16)

4.2. An Algorithm for the Image Processing

In this section, we will explain the application of the previously described quantizers (fixed and adaptive) on grayscale images. Pixels of the image can take integer values from 0 to $x_{max}$. We consider the case whit $x_{max} = 255$ (each pixel is represented with 8 bits). The algorithm for the image processing starts with dividing the image into blocks, whose size is $m \times m$ (we will use $m = 4$). Each block is processed separately. Now, an algorithm will be described, where the fixed piecewise uniform quantizer is used for the processing of one arbitrary block, which is denoted with B and whose pixels are denoted with $x_{i,j}$, $i = 1, \ldots, m$; $j = 1, \ldots, m$.

1. The average value $x_{av}$ of the block B is found, this value is quantized and obtained quantized average value $\hat{x}_{av}$ is transmitted to the receiver.
from values of $PSQNR_{\text{av}}$ (in Fig. 4). It proves that the theory obtained in Section 2 is correct. Also, it justifies the introduction of the weighting function for $\sigma_d$ and shows that $PSQNR_{\text{wav}}$ should be used instead of $PSQNR_{\text{av}}$.

Experimental results for the fixed piecewise uniform quantizer designed for continual input samples are similar with the results in Table 4, since the value of $\sigma_d$ which was used ($\sigma_d = 32$) is adjusted to the test images.

### 5.2. Numerical Results for the Adaptive Piecewise Uniform Quantizer

In this section, numerical results for the adaptive piecewise uniform quantizer for discrete input samples are given. In Fig. 5, theoretically obtained (using the expressions (5), (12) and (13) for the distortion) dependences of $PSQNR$ on $\sigma_d$ [dB] are shown for $N = 16$ and $N = 32$. $PSQNR_{\text{wav}}$ is also shown in this figure. $PSQNR_{\text{av}}$ is not considered, due to the conclusion in the previous section. The optimal values of the parameter $k$, which are numerically found to maximize $PSQNR_{\text{wav}}$, are also given in Fig. 5.

From Fig. 5 we can see an interesting effect: $PSQNR$ for adaptive quantizers increases for small $\sigma_d$ [dB]. Now, we will explain this effect considering one example with parameters $N_0 = 256$, $N = 32$ and $L = 8$. For the fixed quantizer, we choose 32 output levels from the set of 256 input levels, i.e., matching between input and output levels is 12.5%. For the adaptive quantizer, if $\sigma_d$ is small, $x_{\text{max}}$ also will be small. We consider the case:
Fig. 5. PSQNR of adaptive piecewise uniform quantizer for discrete input for: (a) \( N = 16 \) and (b) \( N = 32 \).

Table 5
Experimental results for the adaptive piecewise uniform quantizer designed for discrete and continual input samples

<table>
<thead>
<tr>
<th>( N = 16 )</th>
<th>( N = 32 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L = 2 )</td>
<td>( L = 4 )</td>
</tr>
<tr>
<td>PSQNR [dB]</td>
<td>( 45.029 )</td>
</tr>
<tr>
<td>( \text{PSQNR}_{\text{cont}} ) [dB]</td>
<td>( 41.929 )</td>
</tr>
</tbody>
</table>

\( x_{\text{adapt}} = \frac{1}{4} x_{\text{max}} \). Therefore, inside the range of the adaptive quantizer \((-x_{\text{max}}, x_{\text{max}})\) there are \( N_0/4 = 64 \) input levels and among them we choose \( N = 32 \) output levels. So, for the adaptive quantizer, matching between input and output levels is 50%. If some input level is equal to some output level, distortion for that input level is zero. Therefore, if percentage of matching between input and output levels increases then distortion decreases and then PSQNR increases. It is clear from the above example that matching percentage increases when \( \sigma_d \) decreases.

In Table 5, experimental results for the adaptive piecewise uniform quantizer for discrete input samples are given, for different values of \( N \) and \( L \), obtained by applying this quantizer in Step 3 of the algorithm in Section 4.2. Values in Table 5 are averaged values for standard test images (Lena, Street and Boat). We can see that experimentally obtained values of PSQNR in Table 5 are matched very well to theoretically obtained values of \( \text{PSQNR}_{\text{wav}} \) in Fig. 5. It proves the theory from Section 3.